UDC 62-50

HIERARCHICAL DIFFERENTIAL GAME

PMM Vol. 42, № 1, 1978, pp. 23-27

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(Received March 24, 1977)

A differential game with nonopposing interests and with a fixed order of making decisions by the players is examined. In a game with a fixed sequence of moves the first player first selects a strategy and communicates it to the second player. Having available some information or other on the second player's criterion, the first player can to some extent predict his answering move, evaluating the strategy's effectiveness against the most unfavorable outcome. The least upper bound of the effectiveness evaluations over all strategies of the first player is called the most guaranteed result [1]. The problem of finding the most guaranteed result in static games has been solved under various assumptions on the first player's strategic capabilities and on the information available to him about the second player's criterion (see [2] for the most complete exposition of the theory). A common element of the mathematical techniques for solving these problems is the so-called "punitive strategy" arising from the solution of auxiliary antagonistic games. In the present paper the thoery of position differential games [3] is taken as the base of the theory of antagonistic differentail games; it proves to be convenient to analyze special encounter-evasion problems as the auxiliary antagonistic games; Such an approach permits a unified investigation of several fundamental versions of the first player's strategic capabilities and of his informativeness.

A controlled system described by the following equation of motion

$$dx/dt = f(t, x, u, v); \; x(t_0) = x_0, \; u \in P, \; v \in Q$$

is given. Here $x = x(t)$ is the phase vector, u and v are the first and second player's vector controls of the system and P and Q are compacta. The players strive to maximize their own payoffs which are determined at the game's final position when $t = T$ by continuous functions $g_1(x)$ and $g_2(x)$. The following conditions from [3] are imposed on function $f(t, x, u, v)$: continuity, $\| f(t, x, u, v) \| \leqslant \varkappa (1 + \| x \|)$ (where $\varkappa = \text{const}$) and a Lipschitz condition in the variable x. The first player's moves consist in choosing a certain set G of positions (t, x) and specifying a strategy $U \div u(t, x)$ on it [3]. The pair $\{U, G\}$ is communicated to the second player. The first player's pair $\{U, G\}$ chosen thus restricts the set of possible rounds in the game. If $x[t]$ is a game round for $t_0 \leqslant t \leqslant T$, then there exists an extension $U_G \div u_G(t, x)$ of strategy $u(t, x)$ onto the set of all positions, such that $x[\cdot] \in X[\cdot, U_G]$ ($X[\cdot, U_G]$ is the family of motions [3] from position (t_0, x_0), corresponding to strategy U_G).

Outside set G the second player has the right to deal with not only his own control v but also the first player's control. In particular, let G' be an open set of positions and $G' \cap G = \varnothing$. Then any motion $x[\cdot]$ can be realized as the second player's game match if only $(t, x[t]) \in G'$ for $t_0 \leqslant t \leqslant T$ and $x[t_0] = x_0$. Let us assume that the second player adheres to the following behavior principle: if a method of action exists guaranteeing him a certain amount of b of payoff, then as the game's final position there is realized only a vector x such that $g_2(x) \geqslant b - \beta$ ($\beta > 0$ is a constant characterizing the second player's threshhold of indifference, known to the first player).

Three situations of the first player's informativeness, I_1, I_2 and I_3, on $g_2(x)$ can be examined:

1) function $g_2(x)$ is known exactly,

2) it is known that $g_2(x)$ is some continuous function such that $g_2^-(x) \leqslant g_2(x) \leqslant g_2^+(x)$ is fulfilled for any x, where $g_2^-(x)$ and $g_2^+(x)$ are known continuous functions,

3) $g_2(x)$ is some function from a finite family of continuous functions $\{g_2^\alpha(x), \alpha \in A\}$.

We introduce the concept of the first player's achieved payoff, corresponding to his informativeness. For I_1 we shall consider the set of all final positions

$$M_1(b, c) = \{x \mid \max[g_1(x) - c, \varphi_1(x; b)] \geqslant 0\}$$
$$\varphi_1(x; b) = b - g_2(x) - \beta$$

From the conditions imposed on the controlled system it follows that all the final positions of the game lie in some compactum; therefore, set $M_1(b, c)$ is also a compactum. In what follows, without any stipulation, we always assume that all final positions x are selected from some compactum. By $W_1(b, c)$ we denote the u_*-stable bridge [3] solving the problem of encounter with set $M_1(b, c)$ at $t = T$. Bridge $W_1(b, c)$ is said to be controlled if we can find a motion $x[\cdot]$ from position (t_0, x_0) such that $(t, x[t]) \in W_1(b, c)$ for $t_0 \leqslant t \leqslant T$ and $g_2(x[T]) > b$.

In case I_2 we set

$$M_2(b, c) = \{x \mid \max[g_1(x) - c, \varphi_2(x; b)] \geqslant 0\}$$
$$\varphi_2(x; b) = b - g_2^+(x) - \beta$$

Let $W_2(b, c)$ be a u_*-stable bridge to $M_2(b, c)$. Bridge $W_2(b, c)$ is controlled if we can find a motion $x[\cdot]$ from position (t_0, x_0) such that $(t, x[t]) \in W_2(b, c)$ for $t_0 \leqslant t \leqslant T$ and $g_2^-(x[T]) > b$.

In case I_3 (we denote $b = (b_\alpha, \alpha \in A)$)

$$M_3(b, c) = \{x \mid \max[g_1(x) - c, \varphi_3(x; b)] \geqslant 0\}$$
$$\varphi_3(x; b) = \min_{\alpha \in A}(b_\alpha - g_2^\alpha(x) - \beta)$$

Let $W_3(b, c)$ be a u_*-stable bridge to $M_3(b, c)$. Bridge $W_3(b, c)$ is controlled if for every $\alpha \in A$ we can find $x^\alpha[\cdot]$ from (t_0, x_0) such that

$$(t, x^\alpha[t]) \in W_2(b, c) \text{ for } t_0 \leqslant t \leqslant T \text{ and } g_2^\alpha(x^\alpha[T]) > b_\alpha$$

We say that c is the payoff achievable by the first player in informativeness situation I_j if position (t_0, x_0) is contained in some controlled u_*-stable bridge $W_j(b, c)$ for some b, where $j = 1, 2, 3$.

Theorem. The first player's most guaranteed result in a hierarchical differential game in the case of informativeness I_j equals γ_j, i.e., the least upper bound of the payoffs achievable $(j=1, 2, 3)$.

To prove the theorem we need two lemmas from the theory of position differential games which we quote without proof.

Lemma 1. Let M_1 and M_2 be compact sets and let M_1 be contained in M_2 with a certain neighborhood. If W_1 is a u_*-stable bridge to M_1 and W_2 is a maximal u_*-stable bridge [3] to M_2, then we can find an open set G' contaning W_1, such that

$$(G' \cap \{(t, x) \mid t \leqslant T\}) \subset W_j(b, c)$$

Lemma 2. Let $X[T, U] \subset M$ where M is a compact set, be fulfilled for some strategy U. Then, if $x[\cdot] \in X[\cdot, U]$, then $(t, x[t])$ is contained for $t_0 \leqslant t \leqslant T$ in a maximal u_*-stable bridge W to set M.

Proof of the theorem. Let us show that if c is the achievable payoff in I_j, then for any $\varepsilon > 0$ the first player can guarantee himself a payoff not less than $c-\varepsilon$. By the same token we shall have justified the definition of achievability. We set $\bar{c}=c-\varepsilon$ and $\bar{b}=b+\delta$ $(\bar{b}=(b_\alpha+\delta,\ \alpha \in A)$ for $j=3)$. It is clear that set $M_j(\bar{b}, \bar{c})$ contains $M_j(b, c)$ together with some neighborhood; consequently, by Lemma 1 we can find an open set $G' \supset W_j(b, c)$, such that $(G' \cap \{(t, x) \mid t \leqslant T\}) \subset W_j(\bar{b}, \bar{c})$. By the definition of a controlled u_*-stable bridge there exists a motion $x[\cdot]$ from position (t_0, x_0), for which

$$(t, x[t]) \in W_j(b, c) \quad \text{for} \quad t_0 \leqslant t \leqslant T \quad \text{and} \quad g_2(x[T]) > b$$

(in case I_3 we should replace b by some b_α with $\alpha \in A$). But then $(t, x[t]) \in G'$ for $t_0 \leqslant t \leqslant T$. If the second player gets the possibility of dealing with the first player's control, as he himself would, in the set $W_j(\bar{b}, \bar{c})$ for a sufficiently small $\delta > 0$, then he can realize $x[t]$ as a game round and ensure himself a payoff larger than $\bar{b}$. Because of the assumptions on the nature of the second player's moves, the first player can be certain that the inequality $g_2(x) \geqslant \bar{b}-\beta$ is satisfied for the game's final position. If the first player selects the complement to $W_j(\bar{b}, \bar{c})$ as G and determines therein the strategy $U_e^j \div u_e^j(t, x)$ extremal [3] to bridge $W_j(\bar{b}, \bar{c})$ then he ensures that the final position belongs to set $M_j(\bar{b}, \bar{c})$. If, however, $x \in M_j(\bar{b}, \bar{c})$ and $g_2(x) \geqslant \bar{b}-\beta$, then a payoff not less than $\bar{c}=c-\varepsilon$ to the first player is ensured.

Let us now prove that for any choice of G and U the first player cannot guarantee himself a payoff larger than γ_j in situation I_j. The first player does not know the second player's capability for forming controls. He knows only that any motion $x[\cdot]$, corresponding to every extension of the first player's strategy, can appear as a game round. In particular, the case when the second player can select, by his own arbitrary rule, any game round corresponding to pair $\{U, G\}$ is not excluded. Let U_G be some extension of U. We set

$$b(U_G, g_2) = \max_{x \in X[T, U_G]} g_2(x)$$

Then the least upper bound on the second player's payoffs is

$$b(g_2) = \max_{x \in X[T, U, G]} g_2(x), \qquad X[T, U, G] = [\bigcup_{U_G} X[T, U_G]]$$

It is clear that as the final game position there can appear any vector from the set (excepting, possibly, some limit points)

$$E(g_2) = \{x \in X[T, U, G] \mid g_2(x) \geqslant b(g_2) - \beta\}$$

Moreover, function g_2 itself can be arbitrary from set I_j ; therefore, in situation I_j the first player can limit the set of final game positions only to the set

$$E^j = \bigcup_{g_2 \in I_j} E(g_2), \qquad E^1 = E^1(g_2)$$

$$E^2 = \{x \in X[T, U, G] \mid g_2^+(x) + \beta \geqslant b(g_2^-)\}$$

$$E^3 = \{x \in X[T, U, G] \mid \max_{\alpha \in A} [g_2^\alpha(x) - b(g_2^\alpha) + \beta] \geqslant 0\}$$

Having applied strategy $\{U, G\}$, the first player cannot guarantee himself a payoff larger than c_j, where

$$c_j = \min_{x \in E^j} g_1(x)$$

At the same time, by the definition of $X[T, U, G]$ and of E^j the strategy $u_G(t, x)$ ensures contact with the set

$$M_j = \{x \mid (g_1(x) \geqslant c_j) \text{ or } (\varphi_j(x; b^j) > 0)\}$$

$$(b^1 = b(g_2), \quad b^2 = b(g_2^-), \quad b^3 = (b(g_2^\alpha), \ \alpha \in A))$$

Set $X[T, U_G]$ is closed; therefore, for any $\varepsilon > 0$ we can find $\bar{b}^j$ and $\bar{c}^j$ such that $0 < c_j - \bar{c}_j < \varepsilon$ and $\bar{b}^j < b^j$ and contact with $M_j(\bar{b}^j, \bar{c}_j)$ from position (t_0, x_0) can be effected. Moreover, we can find $\bar{b}^j < b^j$ (for any $\alpha \in A$) in case I_3) such that a motion $x^j[\cdot] \in X[\cdot, U_G]$ exists satisfying the condition $g_2(x^j[T]) > \bar{b}^j$ ($\bar{b}_\alpha^j$ for $j = 3$). But then by Lemma 2 the maximal u_*-stable bridge $W_j(\bar{b}^j, \bar{c}_j)$ solving the problem of encounter with $M_j(\bar{b}^j, \bar{c}_j)$ at instant $t = T$ is controllable and, consequently, $\bar{c}_j$ is an achievable payoff in informativeness situation I_j. The theorem is proved because $\varepsilon > 0$ is arbitrary.

Assume that the problem of encounter with a compact set M^* at instant $t = T$ from position (t_0, x_0) is solvable. Let W^* be the corresponding maximal u_*-stable bridge. By $X\{\cdot, W^*\}$ we denote the aggregate of all motions $x[\cdot]$ such that $x[t_0] = x_0$ and $(t, x[t]) \in W^*$ for $t_0 \leqslant t \leqslant T$. Set $X\{T, W^*\}$ is a section of set $X\{\cdot, W^*\}$ at $t = T$. By the definition of a u_*-stable bridge, $X\{T, W^*\} \subset M^*$. Assume $M = X\{T, W^*\}$ and let W be the maximal u_*-stable bridge solving the problem of encounter with set M at instant $t = T$. We see that

$$M \supset X\{T, M\} \supset X\{T, W^*\} = M$$

Definition. A compact set M is called a controlling set if $X\{T, W\} = M$.

By $\{M\}$ we denote the aggregate of all controlling sets. We define the following auxiliary static $\{M\}$- game. The first player's strategy is the choice of a set $M \in \{M\}$, while that of the second player is the choice of a vector $x \in M$. The second player strives to maximize the function $g_2(x)$ over set M to within β, while the first strives to maximize $g_1(x)$.

From the theorem proved above follows the

Corollary. The first player's most guaranteed results in the hierarchical differential game and in the $\{M\}$-game coincide.

Notes. 1°. Let us assume that at each position (t, x) the first player knows the value of the second player's control v, i.e., can use the counterstrategy $U^v \div u(t, x, v)$ [3]. Then as his strategy in the hierarchical differential game we take the pair $\{U^v, G\}$; the control $u(t, x, v)$ is given, as before, only for $(t, x) \in G$. The changes in the formulation and proof of the theorem reduce to replacing u_*-stability by u-stability [3].

2°. If the information on the second player's criterion is that the first player knows the function $g_2(x, \alpha)$, continuous on $\{x\} \times A$ such that $g_2(x, \alpha) \equiv g_2(x)$ for some $\alpha \in A$, then all the constructions of case I_3 can be extended to this case. Here, however, $b(\alpha)$ is a function continuous on compactum A.

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Translated by N. H. C.